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# AN EFFICIENT TECHNIQUE FOR SOLVING A CLASS OF INFINITE SYSTEMS IN CONTACT PROBLEMS IN THE THEORY OF ELASTICITY*

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In this paper we illustrate the efficiency of a method (see, e.g. /1, 2/) of solving infinite systems of linear algebraic equations of the first kind with singular coefficient matrices, to which many problems in the theory of elasticity and mathematical physics with mixed boundary conditions (see e.g. /3-9/) reduce. The method is based on a knowledge of the behaviour of the solution of the system for large numbers, which may be determined from an analysis of the behaviour of the initial problems at particular points. This enables us to reduce an infinite system to and efficiently-solvable finite system. The method does not require the factorization of functions, it enables us to find the principal component of the solution of infinite systems and also to find explicit particular solutions of the problem at points where the boundary conditions change. This method imposes practically no restrictions on the problem parameters and the computation of the solution does not require large amounts of computer time.

1. Problems in the theory of elasticity with mixed boundary conditions may be reduced using the methods of operational calculus for semi-infinite and bounded regions (strips, layers, cylinders, wedges, cones, rectangles, circular plates, rings, etc.) to the solution of pairs (triples, etc.) of integral equations or series equations.

In particular, we consider a triple series equation /3, 4/ of the form

$$\sum_{k=0}^{\infty} Q_k K(u_k)\, y(u_k, x) = f(x) \quad (c \leqslant x \leqslant a) \tag{1.1}$$

$$\sum_{k=0}^{\infty} Q_k y(u_k, x) = 0 \quad (d \leqslant x \leqslant c,\ a \leqslant x \leqslant b)$$

Here $Q_k$ are the desired variables, $y(u_k, x)$ and $u_k$ are (respectively) a system of eigenfunctions and eigennumbers of a Sturm-Liouville problem for a second-order differential equation in a finite interval (see /3-4/), the nature of the function $K(u)$ is also described in /3-4/.

In special cases of this problem, associated with a specific coordinate system, the functions $y(u_k, x)$ are trigonometric functions, Bessel functions, Legendre functions or other

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known special functions.

Pairs of series (1.1) in which $f(x) = y(i\varepsilon, x)$, $d = c = 0$ (which does not restrict the generality) may be reduced to the study of infinite systems of linear algebraic equations of the form

$$BX = D \tag{1.2}$$

where $B = \{b_{mn}\}$ is a matrix and $X = \{x_n\}$ and $D = \{d_m\}$ are vectors of infinite order. A specific form of system (1.2) may be found in /3-9/. It is a typical feature of system (1.2) that the matrix $B$ may be represented in the form

$$B = A + B_1, \; A = \{(\delta_n - \gamma_m)^{-1}\} \tag{1.3}$$

where $i\delta_n$ and $i\gamma_m$ correspond to a zero and a pole of $K(u)$ (respectively).

In actual cases /8/, as $n \to \infty$

$$\delta_n = \beta n + b \pm ic_1 \ln n + O(n^{-1} \ln n) \tag{1.4}$$
$$\gamma_n = \beta n + g \pm ic_2 \ln n + O(n^{-1} \ln n)$$

and the matrix $A$ contains non-decreasing diagonal elements. The elements of the matrix $B_1 = \{b_{mn}^1\}$ decrease as $m$ and $n$ increase, at least in inverse proportion.

In the solution of contact problems, the distribution function of the contact pressures under a stamp is of practical importance. This is expressed in terms of the infinite system (1.2) by the equation /2-8/

$$q(x) = K^{-1}(i\varepsilon)\, y(i\varepsilon, x) + \sum_{n=1}^{\infty} x_n \frac{y(i\delta_n, x)}{y(i\delta_n, a)} \tag{1.5}$$

2. If, for example, the zone of contact is fixed, there is no friction between the stamp and the elastic body and

$$q(x) \sim \chi (x - a)^{-1/2} \quad (x \to a, \; \chi = \text{const}) \tag{2.1}$$

Taking into account the fact that in the above cases as $n \to \infty$ we have

$$y(i\delta_n, x) = O(e^{\delta_n x}) \quad (c \leqslant x \leqslant a) \tag{2.2}$$

Eq.(2.1) thus enables us to determine the nature of the behaviour of the coefficients $x_n$ as $n \to \infty$, based on the known sum of the series

$$\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} x^k = (1 - x)^{-1/2} \quad (0 \leqslant x < 1) \tag{2.3}$$

Comparing Eqs.(1.5) and (2.1)-(2.3), we obtain

$$x_n = O\left(\frac{(2n-1)!!}{(2n)!!}\right) \quad \text{or} \quad x_n = O\left(\frac{1}{\sqrt{n}}\right) \quad (n \to \infty) \tag{2.4}$$

The above fact enables us to reduce the infinite system (1.2) to a finite system of equations, if we assume that

$$x_n = x_N \frac{(2n-1)!!}{(2n)!!} \quad (n \geqslant N, \quad x_N = \text{const}) \tag{2.5}$$

Then, with a certain error, the infinite system (1.2) becomes equivalent to a system of $N$ equations

$$\sum_{n=1}^{N} b_{mn} x_n + B_m x_N = d_m \quad (m = 1, 2, \ldots, N) \tag{2.6}$$
$$B_m = \sum_{n=N}^{\infty} \frac{(2n-1)!!}{(2n)!!} b_{mn}$$

Here, experiments show that the error in solving the corresponding problems, after replacing system (1.2) by system (2.6), decreases as the value of $N$ increases.

3. It is very difficult to obtain theoretical estimates of the convergence of the method; thus, we will demonstrate its efficiency for a problem in the theory of elasticity with mixed

boundary conditions, which has an exact solution.

We consider a problem in the theory of elasticity concerning smooth displacement along a stamp of an elastic block of infinite length with a rectangular section /10/. We assume that the stamp is symmetrically located between the two boundaries of the block and comes into full contact with the block, where the opposite face of the block is fixed and the side face of the block is free from contact.

This problem reflects all the particular features of other more complicated problems if we solve it using the technique of reduction of series pairs to an infinite system with a singular matrix.

This problem reduces /10/ to the study of pairs of series (1.1) in which

$$y(v_k, x) = \cos u_k x, \; f(x) = 1 \tag{3.1}$$
$$K(u) = u^{-1} \operatorname{th} uh, \; u_k = \pi k/b, \; c = d = 0$$

Here, $h$ is the height of the rectangle, $2b$ is its width, and $2a$ is the width of the stamp.

The elements of the matrix $B_1$ and of the right-hand side $D$ of the infinite system (1.2) were described in /10/.

The distribution of shear stresses under the stamp, according to Eq.(1.5) is given by the formula

$$q(x) = G\delta\left[\frac{1}{h} + \frac{1}{h}\sum_{n=1}^{\infty} x_n \frac{\operatorname{ch}\delta_n x}{\operatorname{ch}\delta_n a}\right] \quad (|x| \leqslant a) \tag{3.2}$$
$$\delta_n = \pi n/h, \; \gamma_m = \pi(2m-1)/(2h)$$

where $G$ is the shear modulus and $\delta$ is the displacement of the stamp.

The connection between the displacement of the stamp and the force $T$ applied to it is given by the equation

$$T = G\delta\left[\frac{2a}{h} + \frac{2}{h}\sum_{n=1}^{\infty}\frac{x_n}{\delta_n} \operatorname{th}\delta_n a\right] \tag{3.3}$$

the matrix and the right-hand side of the infinite system (1.2) of the problem have all the necessary properties described in Sect.1. Thus, in this scheme, the contact stresses and the value of $T$ may be found from the formulae

$$q(x) = \frac{G\delta}{h}\left[1 + \sum_{n=1}^{N-1} x_n \frac{\operatorname{ch}\delta_n x}{\operatorname{ch}\delta_n a} + x_N \sum_{n=N}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{\operatorname{ch}\delta_n x}{\operatorname{ch}\delta_n a}\right] \tag{3.4}$$

$$T = \frac{2G\delta}{h}\left[a + \sum_{n=1}^{N-1} x_n \frac{\operatorname{th}\delta_n a}{\delta_n} + x_n \sum_{n=N}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{\operatorname{th}\delta_n a}{\delta_n}\right] \tag{3.5}$$

where the coefficients $x_n$ $(n = 1, 2, \ldots, N)$ are found from the system of linear algebraic Eqs. (2.6). The main problem as far as numerical implementation of these equations is concerned is the computation of the sum of the weakly converging infinite series in (2.6) and (3.5). The terms of these series behave as $n^{-3/2}$ for large $n$. This difficulty is easy to overcome using a known technique for increasing the convergence of series /11/, which enables us to compute them quickly with high accuracy using finite sums.

For $0 \leqslant x < a$, the series in (3.4) converges like an infinite decreasing geometric progression. For $x = a$, it diverges but the main part of the series in the neighbourhood of $x = a$ is easy to sum and it is thus possible to identify the particular behaviour of the stresses as $x \to a$.

Using the sum of the series (2.3), we obtain an equation for the contact stress in explicit form

$$q(x) = \frac{G\delta}{h}\left[1 + \sum_{n=1}^{N-1}\left(x_n - x_N\frac{(2n-1)!!}{(2n)!!}\right)\frac{\operatorname{ch}\delta_n x}{\operatorname{ch}\delta_n a} - x_N + x_N\left(1 - e^{-\pi(a-x)/h}\right)^{-1/2} + \right. \tag{3.6}$$
$$\left. x_N \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{e^{-2\delta_n x} - e^{-2\delta_n a}}{1 + e^{-2\delta_n a}}\right]$$

From the above expression, we find the singularity coefficient:

$$K_0 = \lim_{x \to a}\sqrt{a^2 - x^2}\, q(x) = G\delta x_N \sqrt{2a/(\pi h)} \tag{3.7}$$

The problem of Sect.3 has an exact solution /12, 13/.

**4.** In order to test the efficiency of this scheme for solving an infinite system (1.2) for the problem of Sect.3, we calculated dimensionless contact stresses $q^*(x) = q(x)\,a/(G\delta)$ of the shearing force $T^* = T/(G\delta)$ and the singularity coefficient $K_0^* = K_0/(\sqrt{2}G\delta)$ from formulae (3.4)-(3.7) as a function of the parameters $\lambda = h/a$ and $\beta = b/a$ and the number of equations $N$ in system (2.6). The calculations showed that this scheme for solving the given model problem converged quite well. To obtain the given accuracy as the parameter $\lambda$ decreases or the parameter $\beta$ increases, the number of equations $N$ is reduced. The nature of the convergence of the method does not depend greatly on the parameters. This enables us to control the accuracy by comparing the results of calculations for various values of the number $N$.

| $N$ | $10^3 \cdot T^*$ | | $10^3 \cdot q^*(0.9)$ | | $10^3 \cdot K_0^*$ | |
|---|---|---|---|---|---|---|
| | $\lambda = 5$ | 10 | 5 | 10 | 5 | 10 |
| 2 | 1280 | 1266 | 942 | 973 | 300 | 334 |
| 6 | 1231 | 979 | 895 | 718 | 273 | 222 |
| 10 | 1231 | 970 | 895 | 708 | 274 | 216 |
| [13] | 1231 | 970 | 894 | 707 | 275 | 218 |

The table shows the values of $T^*$, $q^*(0.9)$ and $K_0^*$ for $\beta = \infty$ together with some values of $\lambda$ and $N$. The last line of the Table shows these values as computed from exact equations in /12, 13/. It is clear that an exact solution of the problem using the above scheme may be efficiently obtained at low cost for $\lambda \leqslant 10$. By increasing the number of equations in system (2.6), the problem may be solved for practically any values of the parameters.

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